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A SOLUTION TO HEAT TRANSFER IN TURBULENT FLOW BETWEEN PARALLEL PLATES

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Abstract—Heat transfer in turbulent flow between parallel plates is solved by matched asymptotic expansion technique for the case of uniform wall temperature, and the Nusselt number and the thermal entry lengths are determined over a wide range of Reynolds and Prandtl numbers. Simple analytical expression is presented for the asymptotic Nusselt number. A comparison of the asymptotic Nusselt number for parallel plates in the range $0.004 < Pr < 0.1$ with that obtained from a circular tube solution based on the equivalent diameter shows that circular tube results if applied for parallel plates overestimate the Nusselt number from 13 to 35% depending upon the Prandtl and Reynolds numbers.

NOMENCLATURE

A_n, constant defined by equation (7);
C_m, constant in equation (3f);
D_e, $= 4h$, equivalent diameter;
$f(\eta)$, $= \dfrac{u}{cu_m}$, dimensionless velocity defined by equation (3b);
F_n, constant defined by equation (6);
G, dimensionless constant defined by equation (13d);
H_n, the nth eigenfunction;
h, one-half the distance between the plates;
J_p, Bessel function of order p;
K, $= G\cdot\lambda$, redefined eigenvalue, equation (13c);
k, thermal conductivity;
m, exponent in the power law velocity;
Nu, Nu_∞, local and asymptotic Nusselt number respectively;
Pr, Prandtl number;
R, redefined eigenfunction given by equation (13a);
Re, Reynolds number;
s, $= KX$, dimensionless stretched radial coordinate;
T, temperature;
T_0, temperature at the inlet ($x = 0$);
u, velocity;
u_m, bulk mean velocity;
v, $= K - s$, stretched distance from the wall;
$w(X)$, function defined by equation (13e);
x, dimensional axial variable;
X, redefined dimensionless radial coordinate defined by equation (13b);
y, dimensional transverse coordinate;
z, $= 1-\eta$, dimensionless distance from the wall.

Greek symbols

α, $= \dfrac{k}{\rho C_p}$, thermal diffusivity;
β_1, constant defined by equation (18);
$\Gamma(P)$, gamma function of argument P;
ε, eddy viscosity;
ε_H, eddy diffusivity;
$\varepsilon(\eta)$, $= 1 + \dfrac{\varepsilon_H}{v'}\cdot Pr$, dimensional total diffusivity;
η, $= y/h$, dimensionless transverse coordinate;
θ, $= \dfrac{T-T_1}{T_0-T_1}$, dimensionless temperature profile;
λ_n, the nth eigenvalue;
μ, $= 1 - X$, dimensionless distance from the wall, equation (18);
v, constant defined by equation (21b);
v', kinematic viscosity;
ξ, dimensionless axial variable defined by equation (3d);
ρ, density;
φ_1, φ_2, constants defined by equation (22c).

Subscripts

b, bulk mean fluid property;
c, center region;
0, channel center;
w, wall region;
1, channel wall.

INTRODUCTION

THE PROBLEM considered here is that of an incompressible, constant property fluid in steady, fully developed turbulent flow inside smooth, straight parallel plate channel. A survey of literature reveals that very little work exists on heat transfer for this type of systems and the existing studies are rather limited in their scopes. In most of these studies the limitations arise from the determination of the eigenvalues and eigenfunctions needed for the solution, because only a few of the eigenvalues and eigenfunctions could be computed with the purely numerical approaches. As a result the solutions are applicable only in the regions away from the inlet and over a very limited range of Prandtl numbers. The other limitation arises from the choice of the eddy diffusivity model. For example, Hatton [1] studied heat transfer in the thermal entrance region with turbulent flow between parallel plates; his analysis is applicable only over a very limited range of Prandtl and Reynolds numbers (i.e. $Pr = 1$ and 10, $Re \simeq 7 \times 10^3$ and 7×10^4).

Some of these difficulties can be alleviated if a combination of an analytical and a numerical approach is used for the analysis. In this approach, the first few eigenvalues and eigenfunctions are computed by purely numerical means, and the remaining are determined by the method of asymptotic technique. Sellars, Tribus and Klein [2] and Dzung [3] studied the case of laminar flow inside circular tubes. Sternling and Sleicher [4] used only a first order analysis to study turbulent flow inside tubes for a uniform wall temperature boundary condition; they have the shortcoming that their asymptotically determined eigenfunctions did not match the computer solutions of Sleicher and Tribus [5]. Recently Sleicher *et al.* [6] and Notter and Sleicher [7] used matched asymptotic expansions to solve the turbulent Graetz problem. One purpose of this paper is to present solutions for heat transfer in turbulent flow between parallel plates for use in engineering applications.

ANALYSIS

Consider heat transfer to an incompressible fluid flowing in steady, fully developed, turbulent flow between two smooth, straight parallel plates with their walls kept at a uniform temperature T_1 and the fluid enters the channel at a uniform and constant temperature T_0.

The energy equation for a steady state, fully developed turbulent flow inside a parallel plate channel is taken in the form

$$u\frac{\partial T}{\partial x} = \frac{\partial}{\partial y}\left[(\alpha+\varepsilon_H)\frac{\partial T}{\partial y}\right] \text{ in } 0 \leqslant y \leqslant h,\ x > 0, \quad (1a)$$

subject to the boundary conditions

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0, \quad (1b)$$

$$T(h, x) = T_1, \quad (1c)$$

$$T(y, 0) = T_0. \quad (1d)$$

Here, x and y are the axial and transverse coordinates, u and T are the axial velocity and temperature, and α and ε_H are the thermal and eddy diffusivities, respectively. The assumption of negligible axial conduction is reasonable when Peclet number exceeds 100.

These equations are now expressed in the dimensionless form as

$$f(\eta)\frac{\partial \theta}{\partial \xi} = \frac{\partial}{\partial \eta}\left[\varepsilon(\eta)\frac{\partial \theta}{\partial \eta}\right] \text{ in } 0 \leqslant \eta \leqslant 1,\ \xi > 0, \quad (2a)$$

with the boundary conditions

$$\left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0} = 0, \quad (2b)$$

$$\theta(1, \xi) = 0, \quad (2c)$$

$$\theta(\eta, 0) = 1, \quad (2d)$$

where various dimensionless variables are defined as

$$\eta = \frac{y}{h}, \quad (3a)$$

$$f(\eta) = \frac{u}{cu_m} = (1-\eta)^{1/m}, \quad (3b)$$

$$\theta = \frac{T-T_1}{T_0-T_1}, \quad (3c)$$

$$\xi = \frac{16x}{cD_e Pr Re}, \quad (3d)$$

$$\varepsilon(\eta) = \frac{\alpha+\varepsilon_H}{\alpha}, \quad (3e)$$

with

$$u = C_m(h-y)^{1/m} \quad (3f)$$

$$c = \frac{1+m}{m}, \quad D_e = 4h, \quad (3g)$$

where u_m is the bulk mean velocity and C_m is a constant. The values of the exponent m and the expression used to define the eddy diffusivity $\varepsilon(\eta)$ are given in the Appendix. A power law velocity distribution is chosen for this study; it will be shown later in this paper that the heat-transfer results obtained from the solution of the energy equation by using a power law velocity profile are in close agreement with those obtained by using the usual logarithmic velocity profile.

Appropriate eigenvalue problem for the solution of equations (2) is given as

$$\frac{d}{d\eta}\left[\varepsilon(\eta)\frac{dH_n(\eta)}{d\eta}\right] + \lambda_n^2 f(\eta)H_n(\eta) = 0, \quad (4a)$$

subject to the boundary conditions

$$H'(0) = 0, \quad (4b)$$

$$H(1) = 0, \quad (4c)$$

with the normalizing condition taken as

$$H(0) = 1, \quad (4d)$$

where H_n and λ_n are the eigenfunctions and eigenvalues respectively and the prime denotes differentiation with respect to η.

Then the temperature distribution is taken in the form

$$\theta(\eta, \xi) = \sum_{n=0}^{\infty} F_n H_n(\eta) e^{-\lambda_n^2 \xi} \quad (5)$$

and the F_n are evaluated by utilizing the boundary condition at $\xi = 0$, which results in the relation,

$$\sum_{n=0}^{\infty} F_n H_n = 1;$$

the orthogonality property of the eigenfunctions leads to the determination of F_n as

$$F_n = -\frac{2}{\lambda_n \dfrac{\partial H(1)}{\partial \lambda_n}} \tag{6}$$

and for convenience in the subsequent analysis we introduce a new constant A_n defined as

$$A_n = -\frac{F_n H_n'(1)}{2}. \tag{7}$$

The heat flux at the channel wall is given by

$$q(\xi) = k\left(\frac{\partial T}{\partial y}\right)_{y=h} = -\frac{8k}{D_e}(T_0 - T_1)\sum_{n=0}^{\infty} A_n e^{-\lambda_n^2 \xi} \tag{8}$$

and the Nusselt number is defined as

$$Nu = \frac{q(\xi) D_e}{k(T_1 - T_b)}, \tag{9}$$

where T_b is the bulk temperature determined from

$$T_b = \frac{\int_0^h Tu\,dy}{\int_0^h u\,dy}. \tag{10}$$

The substitution of velocity and temperature profiles from equations (3b) and (5) into equation (10) yields

$$T_b - T_1 = \frac{2(T_0 - T_1)}{\left(\dfrac{m}{1+m}\right)} \sum_{n=0}^{\infty} \frac{A_n}{\lambda_n^2} e^{-\lambda_n^2 \xi}; \tag{11}$$

then the Nusselt number becomes

$$Nu(\xi) = \frac{4\sum_{n=0}^{\infty} A_n e^{-\lambda_n^2 \xi}}{\left(\dfrac{1+m}{m}\right)\sum_{n=0}^{\infty} \dfrac{A_n}{\lambda_n^2} e^{-\lambda_n^2 \xi}}. \tag{12}$$

Determination of the eigenvalues and the eigenfunctions

Although the first few eigenvalues and eigenfunctions ($n \cong 0$ to 3) can be obtained by purely numerical means, the numerical solutions become less accurate as the value of n increases. On the other hand solutions by analytical means are possible only for the large eigenvalues. Therefore numerical solutions coupled with analytical ones are sought for this problem. In this analytical approach the flow field is divided into two separate regions, the eigenvalue problem is simplified for each of these regions consistent with the physical situations prevailing in them, the resulting equations are then solved analytically and solutions are matched to determine the constants. We present here briefly the numerical and analytical solution of the eigenvalue problem.

Computer calculations of λ_n^2 *and* A_n

The eigenvalue problem given by equations (4) with the velocity and eddy diffusivity profiles as specified previously is solved numerically by performing the calculations in double precision arithmetic and the resulting values of λ_n^2 and A_n are presented in Table 1.

Asymptotic formulas for the higher eigenvalues and eigenfunctions

We seek solution to equations (4) valid for large values of λ_n^2. It is advantageous to change variables in equation (4a) so that it takes a form suitable for finding solutions; new variables are defined as

$$R = [\varepsilon(\eta)\cdot f]^{1/4} H(\eta), \tag{13a}$$

$$X = \frac{1}{G}\int_0^{\eta} \sqrt{[f/\varepsilon(\eta)]}\,d\eta, \tag{13b}$$

$$K = G\lambda, \tag{13c}$$

$$G = \int_0^1 \sqrt{[f/\varepsilon(\eta)]}\,d\eta, \tag{13d}$$

$$w(X) = [\varepsilon(\eta) f]^{-1/4} \frac{d^2}{dX^2}[\varepsilon(\eta) f]^{1/4}. \tag{13e}$$

Equation (4a) then takes the form

$$\frac{d^2 R}{dX^2} + [K^2 - w(X)]R = 0. \tag{14}$$

Equation (14) will now be solved by the matched asymptotic expansion technique by separating the flow field into two regions, namely, the center and the wall regions. For the center region it is sufficiently accurate to take the velocity and diffusivity as constant; then it follows from the definition of $w(X)$ that in the center region

$$w(X) \equiv w_c(X) = 0. \tag{15}$$

Then equation (14) for the center region becomes

$$\frac{d^2 R_c}{ds^2} + K^2 R_c = 0, \quad \text{where } s = KX. \tag{16}$$

For the wall region the eddy diffusivity and eddy viscosity are negligible; the equation becomes

$$\frac{d^2 R_w}{dv^2} + \left(1 - \frac{\beta_1}{v^2}\right) R_w = 0, \tag{17}$$

where

$$v = K\mu = K - s, \quad \beta_1 = \frac{1+4m}{4(1+2m)^2} \quad \text{and} \quad \mu = 1 - X. \tag{18}$$

Solution for the center region

The solution of equation (16) which satisfies the boundary condition $R_c'(0) = 0$ is taken as

$$R_c(s) = D\cos(s) \tag{19}$$

where the unknown coefficient is evaluated from equation (13a) in the neighborhood of the center ($\eta \to 0$ or $s \to 0$); then

$$R_c(s) = (2/\pi)^{1/2} B \cos(s), \tag{20a}$$

where

$$B = (\pi/2)^{1/2} (\varepsilon_0)^{1/4}. \tag{20b}$$

Table 1. Eigenvalues and constants

Prandtl number	Reynolds number	λ_0^2	λ_1^2	λ_2^2	λ_3^2	A_0	A_1	A_2	A_3
0.0	10^4	2.6393	25.421	71.500	140.95	0.95037	0.86528	0.83120	0.81033
0.002	10^4	2.6410	25.440	71.550	141.06	0.95106	0.86569	0.83146	0.81082
	5×10^5	2.7681	26.223	73.734	145.29	1.04697	0.96130	0.92538	0.90581
	10^6	3.0887	29.593	83.725	165.29	1.18726	1.04745	0.99045	0.96306
0.004	10^4	2.6486	25.520	71.789	141.54	0.95408	0.86754	0.83264	0.81134
	10^5	2.7185	25.882	72.754	143.38	1.01002	0.92679	0.89173	0.87131
	5×10^5	3.4218	33.180	94.346	186.63	1.31953	1.12534	1.04438	1.00544
	10^6	4.6243	46.794	135.490	269.68	1.84210	1.38937	1.22597	1.16212
0.01	10^4	2.718	26.26	74.00	145.98	0.9820	0.8845	0.8434	0.8187
	5×10^4	3.036	29.29	82.82	163.55	1.1310	1.0000	0.9424	0.9100
	10^5	3.479	33.89	96.40	190.81	1.3190	1.1229	1.0357	0.9901
	5×10^5	7.264	77.84	230.35	462.72	2.9590	1.9210	1.5761	1.4600
	10^6	12.133	145.06	443.08	897.99	5.1260	2.6300	2.0390	1.9370
0.02	10^4	2.9786	29.041	82.241	162.88	1.08542	0.94565	0.88087	0.84423
	5×10^4	4.2246	42.025	120.579	239.65	1.61296	1.29150	1.14741	1.07434
	10^5	5.6709	58.075	168.923	337.56	2.22433	1.62706	1.38080	1.27087
	5×10^5	15.6970	188.339	577.680	1174.71	6.62082	3.32049	2.48026	2.28883
	10^6	26.9970	369.633	1166.037	2385.52	11.70377	4.61964	4.03403	3.23782
0.04	10^4	3.737	37.41	107.46	214.10	1.3912	1.1075	0.9706	0.9044
	5×10^4	7.143	75.17	221.10	444.19	2.8198	1.9133	1.5478	1.3900
	10^5	10.667	117.46	351.54	710.75	4.3245	2.6054	2.0068	1.7897
	5×10^5	32.389	428.98	1354.39	2774.31	13.9373	5.7109	4.0909	3.7368
	10^6	53.834	838.24	2692.82	5543.48	23.6853	8.2768	5.6676	5.4420
0.1	10^4	5.971	64.64	192.70	390.21	2.3130	1.4650	1.1350	1.0398
	5×10^4	14.897	173.64	530.16	1081.08	6.1000	3.1930	2.2740	1.9563
	10^5	23.667	290.26	899.35	1837.49	9.8878	4.6373	3.2070	2.7269
	5×10^5	69.681	1105.59	3556.27	7332.81	30.5760	11.2600	7.3928	6.5740
	10^6	103.026	2120.43	6915.16	14 278.11	46.0490	17.0099	10.9390	9.9557
0.72	10^4	18.009	294.86	972.91	2002.79	7.4501	2.2046	1.5698	1.8672
	5×10^4	58.716	999.34	3323.12	6949.39	25.212	6.3549	3.6577	3.01362
	10^5	94.189	1753.4	5846.74	12 223.14	40.991	7.3687	5.7207	4.6314
	5×10^5	330.74	6962.7	23 455.99	48 980.29	147.51	31.5617	17.6214	14.6552
	10^6	593.06	13 160.6	44 563.38	93 003.91	266.46	51.8128	29.0679	24.5110
1.0	10^4	21.114	442.63	1500.58	3064.42	8.8326	1.9106	1.5001	1.2389
	5×10^4	72.39	1530.19	5228.63	11 018.46	31.349	5.5137	3.0349	2.653
	10^5	123.06	2729.94	9362.99	19 699.12	53.877	8.8664	4.4104	4.0002
	5×10^5	450.62	10 967.69	37 847.07	79 501.85	201.694	29.25	15.7321	12.659
	10^6	770.00	20 531.6	70 793.21	148 713.49	347.367	52.751	26.774	20.581
10.0	10^4	55.88				23.7679			
	5×10^4	203.56				89.193			
	10^5	364.29				161.095			
	5×10^5	1464.05				660.398			
	10^6	2707.64				1228.925			
10^2	10^4	125.49				53.552			
	5×10^4	490.50				215.241			
	10^5	892.22				395.013			
	5×10^5	3707.94				1674.005			
	10^6	6938.99				3151.93			
10^3	10^4	274.05				117.068			
	5×10^4	1082.84				482.09			
	10^5	2004.9				887.81			
	10^6	15 782.5				7169.4			
10^4	10^4	575.9				246.101			
	5×10^4	2316.66				1016.98			
	5×10^5	17 813.29				8042.80			

Solution for the wall region

The solution of equation (17) is taken as

$$R_w(v) = v^{1/2}[D_1 J_\nu(v) + D_2 J_{-\nu}(v)], \tag{21a}$$

where

$$\nu = \frac{m}{1+2m}, \tag{21b}$$

and D_1 and D_2 are arbitrary constants which are determined by matching the two solutions given by equations (20a) and (21a) for sufficiently large values of K so that the regions of validity of R_c and R_w overlap, we find

$$\frac{D_2}{B} = -\frac{\sin(K-\varphi_1)}{\sin(\pi\nu)} \tag{22a}$$

and

$$\frac{D_1}{B} = \frac{\sin(K-\varphi_2)}{\sin(\pi\nu)} \tag{22b}$$

where

$$\varphi_1 = \frac{\pi}{2}(\tfrac{1}{2}+\nu) \quad \text{and} \quad \varphi_2 = \frac{\pi}{2}(\tfrac{1}{2}-\nu). \tag{22c}$$

Asymptotic eigenvalues

The asymptotic eigenvalues are determined by the requirement that $H(\eta = 1) = 0$. This condition along with equation (13a) gives

$$R_w(v = 0) = 0. \tag{23}$$

Equation (23) when applied to equation (21a) requires that $D_2 = 0$. Hence, equations (22a) and (13c) yield

$$\lambda_n = \frac{n\pi + \varphi_1}{G} \tag{24a}$$

and then this result when combined with equation (22b) gives

$$\frac{D_1}{B} = (-1)^n. \tag{24b}$$

Asymptotic A_n

To evaluate the asymptotic A_n, the expressions $(\partial H/\partial \eta)_{\eta=1}$ and $(\partial H/\partial \lambda)_{\lambda=\lambda_n}$ are required. First we determine H_w (H in the wall region) from R_w. Equation (21a) after utilizing equation (20b) is written as

$$R_w(v) = (\pi v/2)^{1/2}\varepsilon_0^{1/4}\left[\frac{D_1}{B}J_\nu(v) + \frac{D_2}{B}J_{-\nu}(v)\right]. \tag{25a}$$

where

$$v = 2\nu\lambda z^{1/2\nu}, \quad z = 1-\eta. \tag{25b}$$

Now, $H_w(z)$ is determined from equations (13a) and (25) as

$$H_w(z) = (\pi\nu\lambda)^{1/2}\varepsilon_0^{1/4}\left[\frac{D_1}{B}z^{1/2}J_\nu(2\nu\lambda z^{1/2\nu}) + \frac{D_2}{B}z^{1/2}J_{-\nu}(2\nu\lambda z^{1/2\nu})\right]. \tag{26}$$

The derivative

$$\left(\frac{\partial H}{\partial \lambda}\right)_{\substack{\lambda=\lambda_n\\ \eta=1}}$$

is evaluated from equation (26) and the result is

$$\left(\frac{\partial H}{\partial \lambda}\right)_{\substack{\lambda=\lambda_n\\ \eta=1}} = -\frac{(-1)^n\pi^{1/2}\varepsilon_0^{1/4}G(\nu\lambda)^{(1/2)-\nu}}{\sin(\pi\nu)\Gamma(1-\nu)} \tag{27a}$$

and $(\partial H/\partial \eta)_{\eta=1}$ is found as

$$\left(\frac{\partial H}{\partial \eta}\right)_{\eta=1} = -\frac{(-1)^n\pi^{1/2}\varepsilon_0^{1/4}(\nu\lambda)^{(1/2)+\nu}}{\Gamma(1+\nu)}. \tag{27b}$$

Finally, the expression for A_n is obtained from equations (6), (7) and (27) as

$$A_n = \frac{\sin(\pi\nu)(\nu\lambda_n)^{2\nu}\Gamma(1-\nu)}{G\lambda_n\Gamma(1+\nu)}. \tag{28}$$

Equations (24a) and (28) give the asymptotic expressions for λ_n and A_n respectively. These relations are derived on the assumption that λ_n is large and hence they represent the larger eigenvalues and eigenfunctions.

Table 2. The constant G

Pr	Re = 10^4	5×10^4	10^5	5×10^5	10^6
0.0	0.91907				
0.002	0.91871	0.91699	0.91145	0.90617	0.84884
0.004	0.91711	0.91484	0.91179	0.79767	0.66228
0.01	0.90268	0.85217	0.78792	0.50361	0.36128
0.02	0.85412	0.70102	0.58923	0.31423	0.21998
0.04	0.74282	0.51176	0.40344	0.20270	0.14287
0.1	0.54979	0.32526	0.24810	0.12305	0.08800
0.72	0.24564	0.12628	0.09381	0.04661	0.03403
1.0	0.20110	0.10061	0.07369	0.03644	0.02685
10	0.07277	0.03272	0.02375	0.01199	0.00902
100	0.02525	0.01063	0.00782	0.00425	0.00337
1000	0.00832	0.00362	0.00278	0.00181	0.00159
10000	0.00278	0.00140	0.00119	0.00103	0.00102

The parameter G in equations (24a) and (28) is constant for a given Reynolds and Prandtl number. This constant is defined by equation (13d) and its values determined by numerical integration are given in Table 2.

RESULTS AND DISCUSSION

The Nusselt number and entry length calculations can be performed provided that the necessary eigenvalues, λ_n^2, and the constants, A_n, are available. As discussed previously the aim of the analysis is to obtain lower eigenvalues and the corresponding constants from the computer calculations and the higher ones from the asymptotic formulas; in this manner a continuous range of eigenvalues and constants are developed. This procedure, however, is applicable only if the assumptions made in the derivation of the asymptotic formulas have been realized and there exists a domain where the center and wall regions overlap. The asymptotic formulas are found to be in good agreement with the computer calculations whenever such overlapping exists.

The study also has shown that the extent of agreement depends strongly on Prandtl number, but weakly on Reynolds number. That is, the agreement between the computer and the asymptotic solutions has been found to be excellent for the range of Prandtl number below about 0.1, and rather poor for the higher values of Prandtl number. Therefore, in the determination of the local Nusselt number and thermal entry lengths for Prandtl numbers below 0.1, the asymptotic formulas given by equations (24a) and (28) are used to calculate numerical values of λ_n^2 and A_n for values of n greater than those for which computer solutions are available; for higher Prandtl numbers only the computer solutions are used.

The asymptotic Nusselt number is obtainable from equation (12) by taking only the first term in the series, that is

$$Nu_\infty = \frac{4m}{1+m}\lambda_0^2. \tag{29}$$

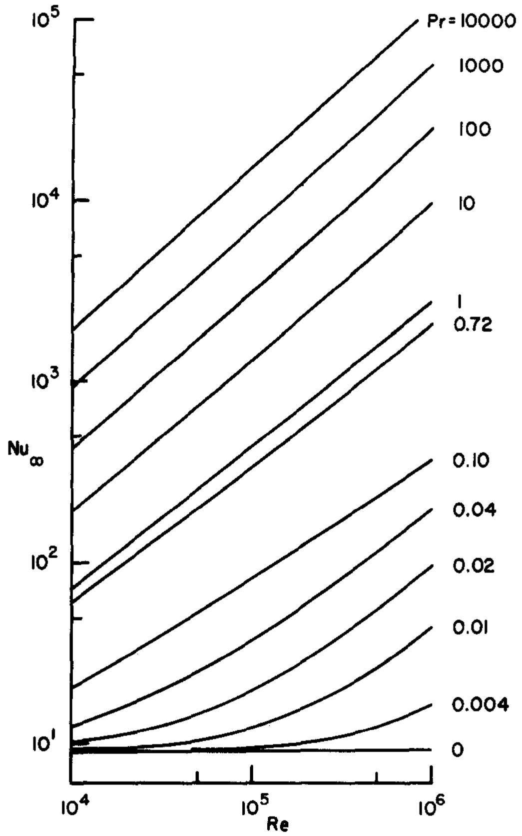

FIG. 1. Effects of Reynolds number on the asymptotic Nusselt number.

Clearly, the asymptotic Nusselt number is directly related to the exponent m and the first eigenvalue, λ_0^2, given respectively in the appendix and Table 1 as a function of Reynolds number and both Reynolds and Prandtl numbers. Values of the asymptotic Nusselt number are shown as a function of Reynolds number for a given Prandtl number in Fig. 1. Figure 2 shows the asymptotic Nusselt number as a function of Peclet number. The asymptotic Nusselt number is correlated to within 6% by the relations

$$Nu = 12+0.03Re^{a_1}Pr^{a_2} \tag{30a}$$

with

$$a_1 = 0.88-\frac{0.24}{(3.6+Pr)} \tag{30b}$$

and

$$a_2 = 0.33+0.5\,e^{-0.6Pr}. \tag{30c}$$

Equations (30) hold for $0.1 < Pr < 10^4$ and $10^4 < Re < 10^6$.

For the low range of Prandtl number the calculations are correlated to within 6% by the relation

$$Nu_\infty = 8.3+0.02Re^{0.82}Pr^{b} \tag{31a}$$

with

$$b = 0.52+\frac{0.0096}{0.02+Pr}. \tag{31b}$$

Equations (31) hold for $0.004 < Pr < 1$ and $10^4 < Re < 10^6$.

In many applications of turbulent flow, the Nusselt number for flow between parallel plates is obtained from those for a circular tube by merely replacing the tube diameter by the effective diameter, D_e, for the flow. For the case of moderate and large Prandtl number, $0.1 < Pr < 10^4$ and $10^4 < Re < 10^6$, a comparison of the asymptotic Nusselt numbers obtained from the present solution for parallel plates with those obtained for circular tubes by Notter and Sleicher [7] based on the effective diameter agreed to within 5%. In the case of low Prandtl number especially in the liquid metals range, however, the use of the circular tube results based on the effective diameter leads to a considerable error in the prediction of the asymptotic Nusselt number. A comparison of the asymptotic Nusselt numbers obtained here for parallel plates in the range $0.004 < Pr < 0.1$ and $10^4 < Re < 10^6$ with those obtained for circular tube by Notter and Sleicher [7] based on the effective diameter shows that the circular tube results modified for parallel plates overestimates the asymptotic Nusselt number by 13–35%, the range of error being dependent upon the Prandtl and Reynolds numbers. The error increases with lower Prandtl numbers, and for a given Prandtl number it increases with higher Reynolds numbers. Therefore

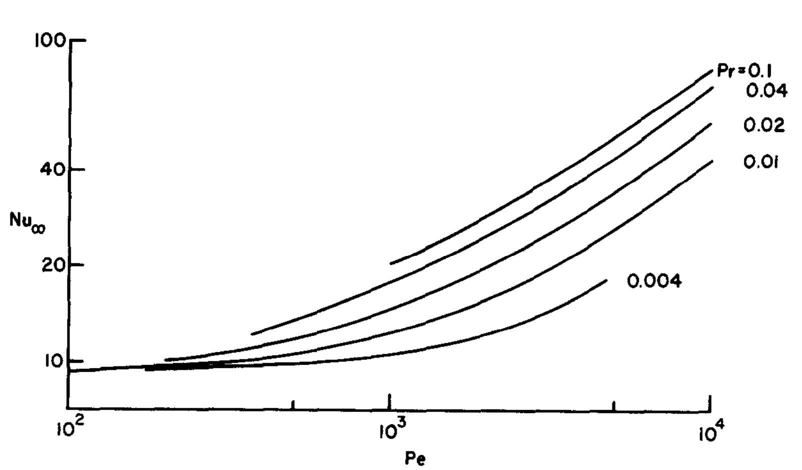

FIG. 2. Asymptotic Nusselt number for liquid metal region.

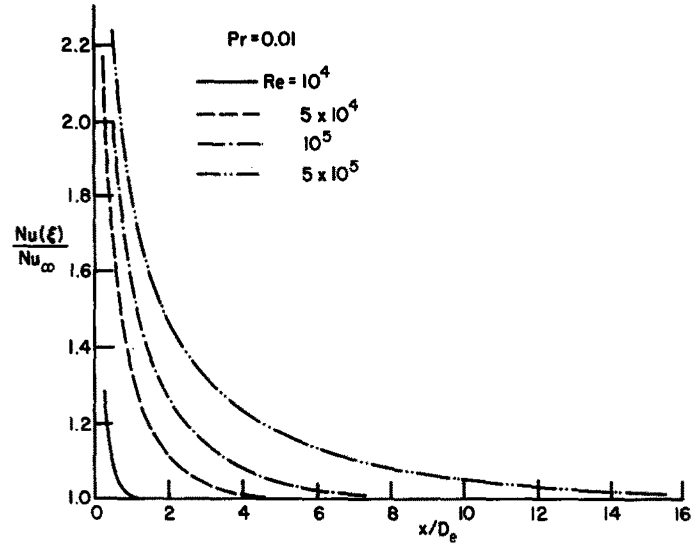

FIG. 3. Effects of Reynolds number on the Nusselt number.

equations (31) are recommended for the calculation of asymptotic Nusselt number for heat transfer to liquid metals in parallel plates.

The local Nusselt number, $Nu(\xi)$, is given by equation (12) and is plotted as a function of X/D_e in Fig. 3.

The thermal entry length is defined in this study to be that distance downstream of the thermal entrance necessary for the local Nusselt number, $Nu(\xi)$, to fall to within 1% of its fully developed value, Nu_∞. Calculations of this quantity, that is x/D_e at which $Nu(\xi)/Nu_\infty = 1.01$, were carried out for $0.002 \leqslant Pr \leqslant 1.0$ and $10^4 \leqslant Re \leqslant 10^6$. The results are shown in Fig. 4.

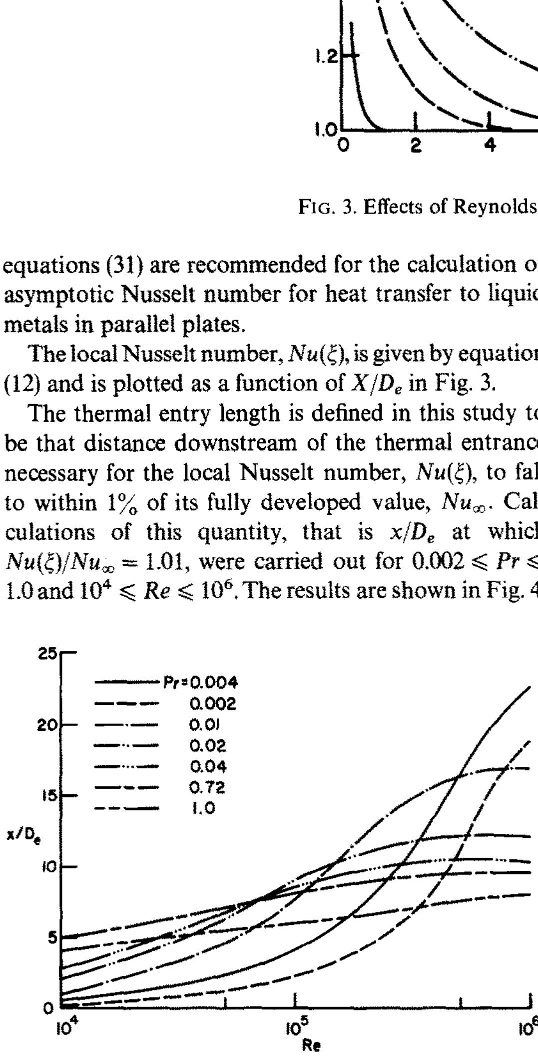

FIG. 4. 1% Thermal entry lengths.

In order to determine the effects of the velocity profile used in the energy equation on the heat-transfer results, the energy equation (2a) is solved by the same technique for the logarithmic velocity distribution used to calculate the eddy diffusivity. Table 3 shows a comparison of the asymptotic Nusselt number for the power law and logarithmic velocity profiles. The two results are sufficiently close to each other.

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Table 3. Effects of the choice of velocity distribution on the Nusselt number

Pr	Re	λ_0^2 Power Law	λ_0^2 Logarithmic profile	$Nu_\infty = \frac{4m}{1+m}\lambda_0^2$ Power Law	$Nu_\infty = 4\lambda_0^2$ Logarithmic profile
0.01	10^4	2.718	2.25669	9.206	9.0267
	5×10^4	3.036	2.6142	10.66	10.4568
	10^5	3.479	3.033	12.32	12.132
	5×10^5	7.264	6.5349	26.23	26.1398
	10^6	12.133	11.04	44.09	44.16
0.1	10^4	5.971	5.01	20.42	20.04
	5×10^4	14.897	12.9812	52.32	51.925
	10^5	23.667	20.86	83.84	83.444
	5×10^5	69.681	63.3028	251.69	253.21
	10^6	103.026	94.419	374.41	377.67
1.0	10^4	21.114	18.0768	72.2	72.307
	5×10^4	72.39	63.836	254.2	255.345
	10^5	123.06	109.529	435.9	438.118
	5×10^5	450.62	410.754	1627.6	1643.016
	10^6	770.00	706.225	2798.0	2824.9

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APPENDIX

The Eddy Diffusivity Profile and the Values of m

The eddy diffusivity profiles

The total diffusivity of heat $\varepsilon(\eta)$ is given in the dimensionless form as

$$\varepsilon(\eta) = 1 + Pr\frac{\varepsilon_H}{v'} = 1 + \frac{Pr}{Pr_t}\cdot\frac{\varepsilon}{v'}. \qquad \text{(A1)}$$

Assuming the shear stress varies linearly with the distance from the wall we write

$$\frac{\varepsilon}{v'} = \frac{\left(\frac{\partial u}{\partial z}\right)_{z=0}\eta}{\frac{\partial u}{\partial z}} - 1. \qquad \text{(A2)}$$

To determine ε/v', the velocity distribution is taken as

$$u^+ = \frac{1}{0.091}\tan^{-1}(0.091Y^+), \quad 0 < Y^+ < 45 \qquad \text{(A3a)}$$

$$u^+ = 5.1 + 2.5\ln Y^+, \quad 45 < Y^+ < \left[\frac{Re}{2}(f_m/8)^{1/2}z\right]_{z=0.15} \qquad \text{(A3b)}$$

$$u^+ = \frac{u_{max}}{u_m}\cdot\frac{1}{(f_m/8)^{1/2}} - h(z), \quad Y^+ > \left[\frac{Re}{2}(f_m/8)^{1/2}z\right]_{z=0.15} \qquad \text{(A3c)}$$

where

$$u^+ = f/(f_m/8)^{1/2},$$

$$Y^+ = \frac{Re}{2}(f_m/8)^{1/2}\cdot z,$$

and the values of u_{max}/u_m are tabulated in Table A1, the friction factor f_m and the velocity defect law $h(z)$ are given as

$$h(z) = 5.75\log(1/z), \qquad \text{(A4a)}$$

$$f_m = 1/\{2\log[Re(f_m)^{1/2}] - 0.8\}^2. \qquad \text{(A4b)}$$

The substitution of the above velocity profiles into equation (A2) gives the expressions for ε/v'.

The turbulent Prandtl number, Pr_t, needed in equation (A1) is taken for $Pr \leqslant 1$ as suggested by Notter and Sleicher [7], as

$$\frac{1}{Pr_t} = \frac{0.025Pr\frac{\varepsilon}{v'} + 90Pr^{3/2}\left(\frac{\varepsilon}{v'}\right)^{1/4}}{1 + 90Pr^{3/2}\left(\frac{\varepsilon}{v'}\right)^{1/4}}\left(1 + \frac{10}{35 + \frac{\varepsilon}{v'}}\right) \qquad \text{(A5)}$$

In the case of Prandtl numbers greater than 1, the total eddy diffusivity $\varepsilon(\eta)$ in the wall region, $0 < Y^+ < 45$, is obtained from equation (A1) by using the relation for ε_H/v' given by Notter and Sleicher [7] as

$$\frac{\varepsilon_H}{v'} = \frac{0.0009Y^{+3}}{(1 + 0.0067Y^{+2})^{1/2}}, \quad 0 < Y^+ < 45. \qquad \text{(A6)}$$

For the regions $Y^+ > 45$, equations (A2) and (A3) are used in such a manner that at $Y^+ = 45$, ε_H/v' from equation (A6) is matched with $(Pr_t\cdot\varepsilon/v')$ from equations (A2) and (A3) to obtain the corresponding turbulent Prandtl number. A similar approach is followed at

$$Y^+ = \left[\frac{Re}{2}(f_m/8)^{1/2}z\right]_{z=0.15}.$$

In the case of the power law velocity profile the values of the exponent m are computed from the relation

$$m = 2/[(1 + 8u_{max}/u_m)^{1/2} - 3] \qquad \text{(A7)}$$

and the results are given in Table A1 as a function of the Reynolds number.

Table A1. Values of $\frac{u_m}{u_{max}}$ and m

Reynolds number	$\frac{u_m}{u_{max}}$	m
10^4	0.788	5.890
5×10^4	0.821	7.198
10^5	0.832	7.748
5×10^5	0.857	9.310
10^6	0.865	9.930

METHODE D'EVALUATION DU TRANSFERT DE CHALEUR EN ECOULEMENT TURBULENT ENTRE PLAQUES PARALLELES

Résumé—Le transfert de chaleur en écoulement turbulent entre plaques parallèles est résolu à l'aide d'une technique de développement asymptotique avec raccordement dans le cas d'une température constante à la paroi; les nombres de Nusselt et les longueurs d'établissement thermique sont déterminés sur une plage étendue de nombres de Reynolds et de Prandtl. Une formule analytique simple est donnée pour le nombre de Nusselt asymptotique. Une comparaison du nombre de Nusselt asymptotique pour l'écoulement entre plaques parallèles dans le domaine $0{,}004 < Pr < 0{,}1$ avec celui en tube circulaire obtenu par une solution basée sur le diamètre équivalent montre que si les résultats en tube circulaire sont appliqués au cas des plaques parallèles on surestime le nombre de Nusselt de 13 à 35 pour cent suivant les valeurs des nombres de Prandtl et de Reynolds.

EINE LÖSUNG FÜR DEN WÄRMEÜBERGANG BEI TURBULENTER STRÖMUNG ZWISCHEN PARALLELEN PLATTEN

Zusammenfassung—Der Wärmeübergang bei turbulenter Strömung zwischen parallelen Platten wird für den Fall einheitlicher Wandtemperatur mit Hilfe einer angepaßten asymptotischen Entwicklung gelöst; die Nusselt-Zahl und die thermische Einlaufstrecke werden für einen großen Bereich von Reynolds- und Prandtl-Zahlen bestimmt. Ein einfacher analytischer Ausdruck für die asymptotische Nusselt-Zahl wird angegeben. Ein Vergleich der asymptotischen Nusselt-Zahl für parallele Platten im Bereich von $0{,}004 < Pr < 0{,}1$ mit derjenigen, die sich unter Verwendung des hydraulischen Durchmessers aus der Lösung für das Rohr ergibt, zeigt, daß die auf parallele Platten angewandte Rohrlösung Nusselt-Zahlen ergibt, welche je nach Prandtl- und Reynolds-Zahlen um 13 bis 35% zu hoch liegen.

РЕШЕНИЕ ЗАДАЧИ О ТЕПЛООБМЕНЕ В ТУРБУЛЕНТНОМ ПОТОКЕ МЕЖДУ ПАРАЛЛЕЛЬНЫМИ ПЛАСТИНАМИ

Аннотация — Задача о теплообмене в турбулентном потоке между параллельными пластинами решается методом асимптотического разложения для случая постоянной температуры стенки при значениях числа Нуссельта и длины теплового начального участка, определяемых в широком диапазоне изменений чисел Рейнольдса и Прандтля. Для асимптотического числа Нуссельта приводится простое аналитическое выражение. Сравнение асимптотического числа Нуссельта для параллельных пластин в диапазоне $0{,}004 < Pr < 0{,}1$ с числом Нуссельта, полученным из решения для круглой трубы с введением эквивалентного диаметра, показывает, что использование результатов, полученных для круглой трубы применительно к параллельным пластинам дает завышенные значения числа Нуссельта на 13–35% в зависимости от чисел Прандтля и Рейнольдса.